Europhotonics : Questions in Quantum Optics

1. Demonstrate the identity :

$$\left[\widehat{A},\widehat{B}\widehat{C}\right]=\left[\widehat{A},\widehat{B}\right]\widehat{C}+\widehat{B}\left[\widehat{A},\widehat{C}\right] \tag{1}$$

2. Use the identity of eq. (1) to evaluate the following expressions : (Rappel : $\widehat{N}_\ell=\widehat{a}_\ell^\dagger\widehat{a}_\ell$ and $\left[\widehat{a}_\ell,\widehat{a}_\ell^\dagger\right]=1$),
   A. $\left[\widehat{a}_\ell,\widehat{N}_\ell\right]$
   B. $\left[\widehat{a}_\ell^\dagger,\widehat{N}_\ell\right]$
   C. $\left[\widehat{a}_\ell,\widehat{a}_\ell^\dagger\widehat{a}_\ell\widehat{a}_\ell^\dagger\right]$

For the following questions, let us consider the special (but common case) where $\left[\widehat{A},\widehat{B}\right]\neq 0$ but we still have the condition that :

$$\left[\widehat{A},\left[\widehat{A},\widehat{B}\right]\right]=0=\left[\widehat{B},\left[\widehat{A},\widehat{B}\right]\right]\ . \tag{2}$$

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3. Under the condition of eq. (2), demonstrate the identity :

$$\left[\widehat{B},\widehat{A}^n\right]=n\widehat{A}^{n-1}\left[\widehat{B},\widehat{A}\right]\ . \tag{3}$$

4. Use the identity derived in eq. (3) to show that :

$$\left[\widehat{B},e^{-\widehat{A}x}\right]=-xe^{-\widehat{A}x}\left[\widehat{B},\widehat{A}\right]\ . \tag{4}$$

5. Use the identity eq. (4) to derive the following expression :

$$e^{\widehat{A}x}\widehat{B}e^{-\widehat{A}x}=\widehat{B}-x\left[\widehat{B},\widehat{A}\right]\ . \tag{5}$$

6. Let us define the operator, $\widehat{O}(x)\equiv e^{\widehat{A}x}e^{\widehat{B}x}$. Calculate the derivative : $\frac{d\widehat{O}}{dx}$, and use the expression of eq. (5) to show that :

$$\frac{d\widehat{O}}{dx}=\left(\widehat{A}+\widehat{B}+x\left[\widehat{A},\widehat{B}\right]\right)\widehat{O}\ . \tag{6}$$

7. Since $\left[\widehat{A},\widehat{B}\right]$ commutes with both $\widehat{A}$ and $\widehat{B}$, we can solve eq. (6) with the usual solution to differential equations methodology. In this manner, derive the relation (known as the disentangling theorem) :

$$e^{\widehat{A}}e^{\widehat{B}}=e^{\widehat{A}+\widehat{B}+\frac{1}{2}\left[\widehat{A},\widehat{B}\right]}\ . \tag{7}$$

8. Let us consider the following harmonic oscillator state (or alternatively a monomode electromagnetic state) :

$$|\Psi\rangle=\frac{1}{\sqrt{2}}\left\{|0\rangle+|4\rangle\right\}\ . \tag{8}$$

   A. Calculate $\overline{n}=\langle\Psi|\,\widehat{N}|\Psi\rangle=\langle\Psi|\,\widehat{a}^\dagger\widehat{a}|\Psi\rangle$.
   B. Calculate $\overline{n^2}=\langle\Psi|\,\widehat{N}^2|\Psi\rangle$.
   C. Calculate $\Delta n\equiv\sqrt{\overline{n^2}-\overline{n}^2}$.

Let us continue to consider monomode radiation state. The quadrature operators of the mode are :

$$X_1 = \frac{1}{\sqrt{2}}\left(\widehat{a} + \widehat{a}^\dagger\right)$$
$$X_2 = \frac{1}{i\sqrt{2}}\left(\widehat{a} - \widehat{a}^\dagger\right) \tag{9}$$

9. For the vacuum state of the monomode state, $|0\rangle$, and the quadrature operators , X_1, and X_2, given in eq. (9) :
   A. Calculate $\langle X_i \rangle_0 = \langle 0| X_i |0\rangle$, for $i = 1, 2$.
   B. Calculate $\langle X_i^2 \rangle_0 = \langle 0| X_i^2 |0\rangle$, for $i = 1, 2$.
   C. Calculate $\Delta X_i = \sqrt{\langle X_i^2\rangle_0 - \langle X_i\rangle_0^2}$ for $i = 1, 2$.

Let us now consider a monomode photon state, $|\Psi\rangle$, which is a superposition of the vacuum, $|0\rangle$ and a 1-photon state, $|1\rangle$. We can then adopt the 'qubit' notation for this state and write an arbitrary superposition as :

$$|\Psi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle\ . \tag{10}$$

where $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi[$. **N.b.** the states $|0\rangle$ and $|1\rangle$ are related by the usual 'ladder' operators $\widehat{a}^\dagger|0\rangle = |1\rangle$, $\widehat{a}|1\rangle = |0\rangle$.

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10. For the state $|\Psi\rangle$ of eq. (10) :
    A. Calculate $\langle X_i \rangle_\Psi = \langle \Psi| X_i |\Psi\rangle$, for $i = 1, 2$
    B. Calculate $\langle X_i^2 \rangle_\Psi \equiv \langle \Psi| X_i^2 |\Psi\rangle$, for $i = 1, 2$.
    C. Calculate $\Delta X_i(\theta, \phi) = \sqrt{\langle X_i^2\rangle_0 - \langle X_i\rangle_0^2}$ for $i = 1, 2$.
    D. Calculate $\Delta X_1 \Delta X_2$ as a function of θ (and an arbitrary value of φ since $\Delta X_1 \Delta X_2$ is φ independent as you can verify), and plot $\Delta X_1 \Delta X_2$ as a function of $\theta \in \{0, 360\}$. Are the Heisenberg uncertanties satisfied ? Explain the behavior of this plot.
    E. Calculate $\Delta X_1(\theta, \phi)$ and $\Delta X_2(\theta, \phi)$ as functions of θ for $\phi = 0$ and $\phi = \pi/2$ with $\theta \in \{0, 360\}$. Can $\Delta X_i(\theta, \phi)$ have values inferior to those found for the vacuum state in the previous question ? Explain the significance of this.

For the the following questions, let us recall that an arbitrary monomode photon state, $|\psi\rangle$, can be written as a superposition of number states (Fock states), $|n\rangle$:

$$|\psi\rangle = \sum_{n=0}^{\infty} C_n\, |n\rangle\ . \tag{11}$$

The quasi-classical state (Glauber state/ coherent state) is given by the superposition :

$$|\alpha\rangle \equiv e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\, |n\rangle\ . \tag{12}$$

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11. Demonstrate that $|\alpha\rangle$ is an eigenstate of the operator $\widehat{a}$.

12. Demonstrate that one cannot construct an eigenstates of the operator $\widehat{a}^\dagger$. In other words, if $\widehat{a}^\dagger|\beta\rangle = \beta|\beta\rangle$ then $|\beta\rangle$ is the null vector, i.e. $|\beta\rangle = 0$ (not to be confused with the vacuum state, $|0\rangle$)

13. For a coherent state,$|\alpha\rangle$.
    A. Calculate the average number of photons : $\overline{n} = \langle \widehat{n} \rangle \equiv \langle \alpha|\, \widehat{a}^\dagger \widehat{a}\, |\alpha\rangle$.
    B. Calculate $\langle X_i \rangle_\alpha = \langle \alpha| X_i |\alpha\rangle$ pour $i = 1, 2$.
    C. Calculate $\langle X_i^2 \rangle_\alpha \equiv \langle \alpha| X_i^2 |\alpha\rangle$, for $i = 1, 2$.
    D. Calculate $\Delta X_i = \sqrt{\langle X_i^2\rangle_\alpha - \langle X_i\rangle_\alpha^2}$ for $i = 1, 2$.

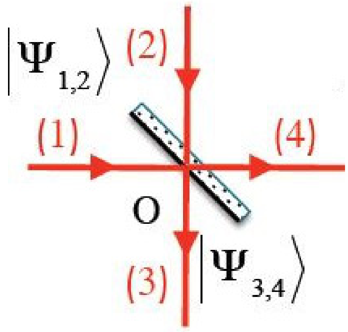

FIGURE 1 – Beam splitter transforming an input representation, $|\Psi\rangle_{1,2}$ into an output representation $|\Psi\rangle_{3,4}$.

For the following questions, let us consider a beamsplitter with an input state, $|\Psi\rangle$, expressed either in the basis of entry channels $|\Psi\rangle_{1,2}$ or in the basis of output channels $|\Psi\rangle_{3,4}$:
One can interpret the beamsplitter as acting on the destruction operators in the following manner :

$$\widehat{a}_3 = r\widehat{a}_1 + t\widehat{a}_2 \qquad \widehat{a}_4 = t\widehat{a}_1 + r\widehat{a}_2 \ ,$$

where r and t are the **complex valued** reflection and transmission coefficients of the beamsplitter (obtained either by measurement or an electromagnetic optics calculation) :

$$\begin{bmatrix} \widehat{a}_3 \\ \widehat{a}_4 \end{bmatrix} = \begin{bmatrix} r & t \\ t & r' \end{bmatrix} \begin{bmatrix} \widehat{a}_1 \\ \widehat{a}_2 \end{bmatrix} = [S] \begin{bmatrix} \widehat{a}_1 \\ \widehat{a}_2 \end{bmatrix} \ , \tag{13}$$

where $[S]$ is called the S-matrix. In the case of a lossless beamsplitter, the S-matrix must be a unitary matrix, *i.e.*

$$S^\dagger.S = S.S^\dagger = \mathbb{I} \ . \tag{14}$$

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14. Show that the lossless condition of eq. (14) requires the 'obvious' energy conservation relations :

$$|r|^2 + |t|^2 = 1 \tag{15a}$$
$$|r'|^2 + |t|^2 = 1 \ . \tag{15b}$$

Interpret the significance of these relations in terms of beam power (intensity).

15. Show that the lossless condition of eq. (14) also requires the 'less obvious' energy conservation relations :

$$r^*t + t^*r' = 0 \tag{16a}$$
$$t^*r + tr'^{,*} = 0 \ . \tag{16b}$$

Explain why these 'two' conditions are in fact only a single condition on the complex coefficients.

16. For a beam splitter that is mirror symmetric with respect to a plane at the center of the beam splitter's diagonal, often called a *symmetric* beam splitter, one has $r = r'$. For this symmetric case, what do the relations of eq. (16) have to say about the phase relations between the coefficients r and t ? (Hint : write $r = |r|e^{i\phi_r}$ and $t = |t|e^{i\phi_t}$, and determine the relation between ϕ_r and ϕ_t imposed by eq. (16).

17. Which of the following S-matrices are physically acceptable for a lossless beam splitter ?
    - A. $[S] = \begin{bmatrix} i\rho & t \\ t & i\rho \end{bmatrix}$ for real-valued ρ and t with $\rho^2 + t^2 = 1$.
    - B. $[S] = \begin{bmatrix} r & t \\ t & -r \end{bmatrix}$ for r and t real-valued with $r^2 + t^2 = 1$.
    - C. $[S] = \begin{bmatrix} \rho & i\tau \\ -i\tau & \rho \end{bmatrix}$ for ρ and τ real-valued with $\rho^2 + \tau^2 = 1$.
    - D. $[S] = \begin{bmatrix} r & i\tau \\ i\tau & r \end{bmatrix}$ for r and τ real-valued with $r^2 + \tau^2 = 1$ for ρ and τ real with $r^2 + \tau^2 = 1$

18. Given the S-matrix relation of eq. (13), and the results of previous questions, show that transformation of the creation operators is :

$$\begin{bmatrix}\hat{a}_3^\dagger \\ \hat{a}_4^\dagger\end{bmatrix} = \begin{bmatrix} r & t \\ t & r' \end{bmatrix} \begin{bmatrix}\hat{a}_1^\dagger \\ \hat{a}_2^\dagger\end{bmatrix} = [S] \begin{bmatrix}\hat{a}_1^\dagger \\ \hat{a}_2^\dagger\end{bmatrix} . \tag{17}$$

19. Let us consider the state $|n\rangle_1$ with n photons in channel 1. Express $|n\rangle_1$ in terms of $\hat{a}_1^\dagger$ and the vacuum state of channel 1, $|0\rangle_1$:
    - A. $|n\rangle_1 = \frac{1}{\sqrt{n!}} \left(\hat{a}_1^\dagger\right)^n |0\rangle_1$
    - B. $|n\rangle_1 = \frac{1}{\sqrt{(n-1)!}} \left(\hat{a}_1^\dagger\right)^n |0\rangle_1$
    - C. $|n\rangle_1 = \sqrt{n!} \left(\hat{a}_1^\dagger\right)^n |0\rangle_1$

20. Express $|\Psi\rangle = |n\rangle_1 \otimes |m\rangle_2 = |n, m\rangle_{1,2}$ in terms of $\hat{a}_1^\dagger$ and $\hat{a}_2^\dagger$ acting the vacuum of the 2 channels $|0, 0\rangle$. Note that the vacuum state doesn't depend on the chosen basis. (more than one correct response possible)
    - A. $|\Psi\rangle = \frac{1}{\sqrt{n!m!}} \hat{a}_1^{\dagger n} \hat{a}_2^{\dagger m} |0, 0\rangle$
    - B. $|\Psi\rangle = \frac{1}{\sqrt{n!m!}} \hat{a}_1^{\dagger m} \hat{a}_2^{\dagger n} |0, 0\rangle$
    - C. $|\Psi\rangle = \frac{1}{\sqrt{n!m!}} \hat{a}_2^{\dagger m} \hat{a}_1^{\dagger n} |0, 0\rangle$

A beamsplitter that is mirror symmetric with respect to its central plane can be written, with r and t being complex valued, and the energy conservation relations is :

$$[S] = \begin{bmatrix} r & t \\ t & r \end{bmatrix} . \tag{18}$$

21. Express the state $|\Psi\rangle = |n, m\rangle_{1,2}$ in terms of the creation operators in the output channels, $\hat{a}_3^\dagger$ and $\hat{a}_4^\dagger$ for a beamplitter described by the mirror symmetric beamsplitter of eq. (18).
    - A. $|\Psi\rangle = \frac{1}{\sqrt{n!m!}} \left(r^*\hat{a}_3^\dagger - t^*\hat{a}_4^\dagger\right)^n \left(t^*\hat{a}_3^\dagger + r^*\hat{a}_4^\dagger\right)^m |0, 0\rangle$
    - B. $|\Psi\rangle = \frac{1}{\sqrt{n!m!}} \left(r^*\hat{a}_3^\dagger + t^*\hat{a}_4^\dagger\right)^n \left(t^*\hat{a}_3^\dagger + r^*\hat{a}_4^\dagger\right)^m |0, 0\rangle$
    - C. $|\Psi\rangle = \frac{1}{\sqrt{n!m!}} \left(r\hat{a}_3^\dagger + t\hat{a}_4^\dagger\right)^n \left(t\hat{a}_3^\dagger + r\hat{a}_4^\dagger\right)^m |0, 0\rangle$

22. Consider the state $|\Psi\rangle = |1, 0\rangle_{1,2}$. Give the expression for this state in the base of the output channels.
    - A. $|\Psi\rangle = |1, 0\rangle_{1,2} = r^*|1, 0\rangle_{3,4} + t^*|0, 1\rangle_{3,4}$
    - B. $|\Psi\rangle = |1, 0\rangle_{1,2} = t^*|1, 0\rangle_{3,4} + r^*|0, 1\rangle_{3,4}$
    - C. $|\Psi\rangle = |1, 0\rangle_{1,2} = r|1, 0\rangle_{3,4} + t|0, 1\rangle_{3,4}$

23. Consider the state, $|\Psi\rangle = |1, 1\rangle_{1,2}$, corresponding to exactly 1-photon in each entree mode channel on a mirror symmetric beamsplitter. What is the correct expression for $|\Psi\rangle = |1, 1\rangle_{1,2}$ in the output basis ?
    - A. $|\Psi\rangle = \sqrt{2}\, rt|2, 0\rangle_{3,4} + \left[t^2 + r^2\right] |1, 1\rangle_{3,4} + \sqrt{2}\, tr|0, 2\rangle_{3,4}$
    - B. $|\Psi\rangle = \sqrt{2}\, r^*t^*|2, 0\rangle_{3,4} + \left[(t^*)^2 + (r^*)^2\right] |1, 1\rangle_{3,4} + \sqrt{2}\, t^*r^*|0, 2\rangle_{3,4}$
    - C. $|\Psi\rangle = r^*t^*|2, 0\rangle_{3,4} + \left[(t^*)^2 + (r^*)^2\right] |1, 1\rangle_{3,4} + t^*r^*|0, 2\rangle_{3,4}$

24. Let us continue with the preceding question with the state $|\Psi\rangle = |1, 1\rangle_{1,2}$ corresponding to exactly a photon in each entry channel. Consider now a symmetric 50/50 beamsplitter with $r = \frac{i}{\sqrt{2}}$ and $t = \frac{1}{\sqrt{2}}$. What is the probability to detect precisely 1 photon in each output mode channel ? (This is the famous Hong-Ou-Mandel effect)
    - A. 0
    - B. 1/3
    - C. 1/4
    - D. 1

25. Consider a partially silvered beam splitter described by the S-matrix in 17.B with $r = t = \frac{1}{\sqrt{2}}$. Demonstrate that this beam splitter generates the same coalesced 2-photon states as obtained with the symmetric beam splitter.